

Cosmology and the Korteweg-de Vries Equation

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The Korteweg-de Vries (KdV) equation is a non-linear wave equation that has played a fundamental role in diverse branches of mathematical and theoretical physics. In the present paper, we consider its significance to cosmology. It is found that the KdV equation arises in a number of important scenarios, including inflationary cosmology, the cyclic universe, loop quantum cosmology and braneworld models. Analogies can be drawn between cosmic dynamics and the propagation of the solitonic wave solution to the equation, whereby quantities such as the speed and amplitude profile of the wave can be identified with cosmological parameters such as the spectral index of the density perturbation spectrum and the energy density of the universe. The unique mathematical properties of the Schwarzian derivative operator are important to the analysis. A connection with dark solitons in Bose-Einstein condensates is briefly discussed.

PACS numbers: 98.80.Cq

I. INTRODUCTION

The Korteweg-de Vries (KdV) equation [1] is the completely integrable, third-order, non-linear partial differential equation (PDE):

$$\partial_t u + \partial_x^3 u + \frac{3}{u_0} u \partial_x u = 0, \quad (1)$$

where $u = u(x, t)$, $\partial_t = \partial/\partial t$, $\partial_x^3 = \partial^3/\partial x^3$, etc., u_0 is a constant and (x, t) represent space and time coordinates, respectively. This equation was originally derived within the context of small-amplitude, non-linear water wave theory and it is well known that it admits a solitonic wave solution of the form

$$u = u_0 \lambda^2 \text{sech}^2[\lambda(x - \lambda^2 t)/2], \quad (2)$$

where the constant $\lambda/2$ represents the wavenumber of the soliton [1]. The KdV soliton is characterized by the property that its speed and amplitude are proportional to the square of the wavenumber.

The KdV equation has played a central role in diverse branches of physics, including nonlinear optics, atomic and nuclear physics, Bose-Einstein condensates and astrophysical plasmas (see Ref. [2] for a review.) As far as we are aware, however, it has not been discussed previously within a cosmological context (although see Ref. [3]). This is perhaps not surprising, given that the KdV equation is a third-order PDE in two independent variables, whereas the field equations for spatially isotropic universes are second-order, ordinary differential equations (ODEs).

One the other hand, Eq. (1) can be reduced to the non-linear ODE:

$$-\lambda^2 u' + u''' + \frac{3}{u_0} u u' = 0, \quad (3)$$

where a prime denotes $d/d\phi$ and $\phi \equiv x - \lambda^2 t$ represents a ‘wave-like’ independent variable. The purpose

of the present paper is to show that wave solutions to Eq. (3), and in particular the soliton solution (2), arise in a number of cosmological settings, including the inflationary paradigm, the cyclic universe scenario, loop quantum cosmology and braneworld models. We consider scenarios where the universe is dominated by a single, self-interacting scalar field, ϕ . Interpreting the value of the scalar field in terms of a wave-like coordinate of the KdV equation then allows for a direct analogy to be drawn between cosmological dynamics on the one hand and solitonic behaviour on the other. Cosmological parameters can then be identified with quantities such as the speed and amplitude of the corresponding wave.

The paper is organised as follows. In Section 2, we discuss a connection between the KdV equation and the Schwarzian derivative operator that we employ in later Sections. We proceed in Section 3 to consider the classes of inflationary and cyclic universes that generate density perturbation spectra with a constant spectral index. In Section 4, we find that the scaling solutions of various braneworld and loop quantum cosmological scenarios are analogous to the KdV soliton. We conclude with a discussion in Section 5 on a connection with dark solitons in Bose-Einstein condensates. Unless otherwise stated, units are chosen such that $\hbar = c = 1$ and the Planck mass is normalised to $m_P = \sqrt{8\pi}$.

II. THE KDV EQUATION AND THE SCHWARZIAN DERIVATIVE OPERATOR

The KdV equation (3) admits an auto-Bäcklund transformation, whereby a solution $u = u(\phi)$ can be derived from a given seed solution $\bar{u} = \bar{u}(\phi)$ [4]. For the special case where the seed is the trivial solution $\bar{u} = 0$, such a transformation reduces to the condition that a solution to Eq. (3) is given by

$$u = u_0(\lambda^2 - 4y^2), \quad (4)$$

where λ^2 is a constant and the function $y = y(\phi)$ is a particular solution to the first-order Riccati equation:

$$y' = \frac{u}{4u_0} = \frac{\lambda^2}{4} - y^2. \quad (5)$$

For example, the solution

$$y = \frac{\lambda}{2} \tanh(\lambda\phi/2) \quad (6)$$

generates the KdV soliton (2).

Solutions to Eq. (5) satisfy a third-order ODE given by

$$S[y(\phi)] \equiv \frac{y'''}{y'} - \frac{3}{2} \left(\frac{y''}{y'} \right)^2 = -\frac{\lambda^2}{2}, \quad (7)$$

as may be verified by direct differentiation of Eq. (5). The left-hand side of Eq. (7) is the Schwarzian derivative operator (often referred to simply as the *Schwarzian*) [5]. This operator exhibits a number of remarkable properties, one of which we exploit in the present work: it is the unique combination of derivatives that is invariant under a homographic transformation corresponding to the group of fractional linear transformations. This follows since, for a function $y = y(\phi)$, the composition $N(y) \equiv [\ln(y')] = y''/y'$ transforms under the inversion $y \rightarrow 1/y$ such that $N(1/y) = N(y) - 2y'/y$. Some straightforward algebra then implies that the operator $S(y) \equiv N' - N^2/2$ is invariant under inversion. Moreover, since $S(y) = S(my + n)$ for any $m, n \in \mathbb{R}$, $S(y)$ is invariant under the full group of fractional linear transformations.

This implies that if $\bar{y}(\phi)$ is a particular ‘seed’ solution to the differential equation $S[y] = f(\phi)$ for some function $f(\phi)$, the *general solution* to such an equation is given by

$$y(\phi) = \frac{a\bar{y} + b}{c\bar{y} + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{R}), \quad (8)$$

where a, b, c, d are constants such that $ad - bc = 1$.

To summarize, therefore, if a particular solution to the Schwarzian equation (7) for constant λ can be found, the general solution (8) can be written down immediately. Restricting the general solution to satisfy the first-order constraint (5) then generates a solution (4) to the KdV equation (3). We employ this result in the following Sections for a number of cosmological models.

III. COSMOLOGICAL DENSITY PERTURBATIONS AND THE KDV SOLITON

A. The Inflationary Universe

The inflationary scenario remains the cornerstone of modern, early universe cosmology. Whilst there is currently considerable interest in multiple-field versions of

the paradigm, our aim in this Section is to revisit the simplest version of the scenario, namely inflation driven by a single, minimally coupled, slowly-rolling scalar inflaton field, ϕ . (For reviews, see, e.g., Ref. [6, 7].) We consider the general class of models that generate a *scale-invariant* density perturbation spectral index (hereafter referred to as the spectral index) to lowest-order in the slow-roll approximation. Our main aim is to highlight some interesting mathematical features of the underlying differential equations that have not been previously discussed.

The cosmological Friedmann equations in Hamilton-Jacobi form are given by

$$3H^2 - 2H'^2 = V(\phi), \quad \dot{\phi} = -2H', \quad (9)$$

where the Hubble parameter $H = H(\phi)$ is viewed as a function of ϕ , $V(\phi)$ denotes the inflaton potential and a prime and dot denote differentiation with respect to the inflaton field and cosmic time, respectively. The energy density of the universe is $\rho(\phi) = 3H^2(\phi)$. It is assumed implicitly and without loss of generality that the inflaton varies monotonically with cosmic time such that $\dot{\phi} > 0$ ($H' < 0$).

The Hubble slow-roll parameters are defined by

$$\epsilon \equiv 2 \frac{H'^2}{H^2}, \quad \eta \equiv 2 \frac{H''}{H}, \quad (10)$$

and the spectral index is given by

$$1 - n_s = \frac{2}{(1 - \epsilon)^2} \left[\epsilon - \frac{(1 - \epsilon^2)}{2} \left(\frac{d \ln \epsilon}{dN} \right) \right], \quad (11)$$

where higher-order derivative terms in $\mathcal{N} \equiv -\ln(aH)$ have been neglected. To lowest-order in the slow-roll approximation, $1 - n_s = 4\epsilon - 2\eta$. This condition is a second-order, non-linear ODE for the dependent variable $H = H(\phi)$:

$$4 \frac{H''}{H} - 8 \frac{H'^2}{H^2} = -(1 - n_s). \quad (12)$$

Considerable insight into the nature of the general solution to Eq. (12) may be gained by expressing the energy density of the universe in terms of the gradient of a ‘potential’ function, $W = W(\phi)$, such that

$$H^2(\phi) \equiv 4H_0^2 W'(\phi), \quad (13)$$

where H_0 is an arbitrary constant. Substituting this definition into Eq. (12) yields

$$S[W(\phi)] \equiv \frac{W'''}{W'} - \frac{3}{2} \left(\frac{W''}{W'} \right)^2 = -\frac{\lambda^2}{2}, \quad (14)$$

where $\lambda^2 \equiv 1 - n_s$.

The left-hand side of Eq. (14) is the Schwarzian derivative of the function $W(\phi)$ and it is interesting that it arises in this cosmological context. Consequently, we may

now determine from Eqs. (13) and (14) the general forms of the Hubble parameters for the full family of inflationary cosmologies that generate a constant spectral index.

Since current observational bounds on the spectral index inferred from the WMAP7+H0 data set are $0.939 < n_s < 0.987$ at the 2σ confidence limit [8], we focus on the red perturbation spectrum, $\lambda^2 > 0$. A particular solution to Eq. (14) is $\bar{W}(\phi) = \exp(\lambda\phi)$ and the general solution is therefore given directly by

$$\begin{aligned} W(\phi) &= \frac{ae^{\lambda\phi} + b}{ce^{\lambda\phi} + d} \\ H^2(\phi) &= 4\lambda H_0^2 \frac{e^{\lambda\phi}}{(ce^{\lambda\phi} + d)^2}. \end{aligned} \quad (15)$$

Moreover, comparing Eq. (14) with Eq. (7) and Eq. (13) with Eq. (5) immediately implies that $H^2(\phi)$ satisfies the KdV equation

$$-(1 - n_s)H^{2'} + H^{2'''} + \frac{3}{H_0^2}H^2H^{2'} = 0 \quad (16)$$

if the general solution to Eq. (14) is restricted to satisfy the Riccati equation

$$W' = \frac{\lambda^2}{4} - W^2. \quad (17)$$

It may be verified that condition (17) is satisfied if

$$ad = -bc = 1/2, \quad \lambda = \frac{1}{cd} = \frac{2a}{c}. \quad (18)$$

As a result, the solution (15) can be expressed as

$$H^2(\phi) = H_0^2 \lambda^2 \operatorname{sech}^2 \left(\frac{\lambda \sqrt{8\pi}}{2} \phi \right) \quad (19)$$

for $cd > 0$ and

$$H^2(\phi) = -H_0^2 \lambda^2 \operatorname{cosech}^2 \left(\frac{\lambda \sqrt{8\pi}}{2} \phi \right) \quad (20)$$

if $cd < 0$, where we have specified $c = |d|$ without loss of generality and have restored the dependence on the Planck mass for future reference. (If $c \neq |d|$, Eqs. (19)-(20) can be recovered by performing a linear translation $\lambda^{-1} \ln |d/c|$ on the value of the inflaton field.)

Given the general form of the Hubble parameter, the inflationary potentials can be deduced directly from the Hamilton-Jacobi equation (9). It is straightforward to verify that these potentials correspond precisely to those derived in [9] to lowest-order in slow-roll. From the cosmic dynamical systems point of view, the late-time attractor of these models is the power-law solution $\epsilon = \lambda^2$. There are two different potentials depending on whether the initial value of ϵ is greater or less than λ^2 [9]. Within the context of the present discussion, the power-law model is the seed solution $\bar{W}(\phi) = \exp(\lambda\phi)$ and the two models are characterized by $\operatorname{sgn}(cd)$, i.e., by $\operatorname{sgn}(W')$.

Eqs. (19) and (20) are both wave solutions to the KdV equation (16) and the former has precisely the form of the non-singular KdV soliton (2). This suggests a direct analogy can be drawn between such a wave and inflationary cosmology. In such an analogy, the inflaton field plays the role of a characteristic, wave-like coordinate on a two-dimensional spacetime $\{x, t\}$, the speed of the soliton is determined by the deviation of the spectral index away from the scale-invariant, Harrison-Zel'dovich spectrum $n_s = 1$, and the amplitude profile of the soliton is parametrized by the energy density of the universe (in appropriate units).

The first slow-roll parameter for the ‘soliton’ solution (19), is given by

$$\epsilon(\phi) = \frac{\lambda^2}{2} \tanh^2(\lambda\phi/2) \quad (21)$$

and determines the tensor-scalar ratio, $r \equiv \mathcal{P}_T^2/\mathcal{P}_S^2 = 16\epsilon$, where $\mathcal{P}_S^2 = H^2/(64\pi^4\epsilon)$ and $\mathcal{P}_T^2 = H^2/(4\pi^4 m_P^2)$ are the amplitudes of the density and gravitational wave perturbations [6, 7]. This parameter is bounded from above such that $r < 8(1 - n_s)$, which is consistent with the current observational 2σ bound $r < 0.24$ [8].

B. The Cyclic Universe

During inflation, quantum fluctuations in the inflaton field become frozen on super-Hubble radius scales because the comoving Hubble scale decreases with time due to the rapid, accelerated expansion of the universe. However, the comoving Hubble radius can also decrease if the universe undergoes a phase of slow, decelerated *contraction* driven by a negative scalar field potential. This is the basis of the cyclic universe scenario. As shown in [10], the expression (11) for the spectral index is invariant under the duality $\epsilon \rightarrow 1/\epsilon$ and this implies there exists a one-to-one correspondence between inflationary and cyclic models that generate identical spectral indices.

This suggests that a similar gravity/soliton analogy may be established for the cyclic universe scenario. Indeed, in the Hamilton-Jacobi formalism of the cosmological Friedmann equations (9), the definition of the Hubble parameter $H = \dot{a}/a$ in terms of the scale factor a implies that

$$a'H' = -\frac{1}{2}aH. \quad (22)$$

Integrating Eq. (22) then yields the dependence of the scale factor on the scalar field in terms of the quadrature

$$a(\phi) = a_0 \exp \left[-\frac{1}{2} \int^\phi d\phi \frac{H}{H'} \right], \quad (23)$$

where a_0 is an arbitrary constant.

However, Eq. (22) is invariant under the simultaneous interchange $H(\phi) \leftrightarrow a(\phi)$. If we therefore consider a

‘dual’ cosmology where the Hubble parameter is given by $\tilde{H}(\phi) = a(\phi)$, Eq. (23) implies that the dual scale factor is given by the quadrature

$$\tilde{a}(\phi) = \tilde{a}_0 \exp \left[-\frac{1}{2} \int^\phi d\phi \frac{a}{a'} \right]. \quad (24)$$

Since the seed cosmology $\{H(\phi), a(\phi)\}$ itself satisfies Eq. (22), the duality between the two scenarios is given by the simultaneous interchange of the Hubble parameters and scale factors of the two scenarios when all parameters are expressed as functions of the scalar field [11]:

$$\tilde{a}(\phi) = H(\phi), \quad \tilde{H}(\phi) = a(\phi). \quad (25)$$

It is straightforward to verify that under this duality, the Hubble slow-roll parameter indeed transforms as $\tilde{\epsilon} = 1/\epsilon$.

As a result of this duality, the analysis of Section 3.1 applies directly to the cyclic universe. We conclude, therefore, that there exists a one-to-one correspondence between solutions to the KdV equation and the respective cyclic cosmological models when the spectral index is constant. For the cyclic model that is dual to Eq. (19), the speed of the soliton is once more determined by the spectral index, whereas the amplitude of the soliton is now proportional to the square of the cosmic scale factor.

IV. MODIFIED COSMOLOGY

In recent years, considerable interest has focused on cosmological dynamics arising from modified gravity theories. This is motivated in part by open questions in early universe cosmology, such as the singularity problem, as well as providing alternative scenarios to dark energy models. At a phenomenological level, such modifications can be quantified by altering the standard form of the Friedmann equation such that

$$3H^2 = \rho L^2(\rho), \quad (26)$$

where $L = L(\rho)$ is a given function of the energy density and is determined by the specific model in question.

Of particular interest in such scenarios are scaling (attractor) solutions, since these enable the generic asymptotic behaviour of a cosmological model to be better understood. Scaling solutions are characterized by the property that the energy densities of the component matter fields scale at the same rate as the universe expands (contracts). Such solutions were classified in Ref. [12] under the assumption that the matter content of the universe is comprised of a self-interacting scalar field with potential $V(\phi)$ and a barotropic fluid with an adiabatic index γ . By introducing a parameter

$$\lambda \equiv -\frac{1}{L} \frac{V'}{V} \quad (27)$$

it was found that $\lambda = \text{constant}$ is a necessary condition for a scaling solution to exist. In that case, there exists an attractor solution for $\lambda^2 > 3\gamma$, where the relative contribution of the scalar field energy density to the total density of the universe is $\Omega_\phi = 3\gamma/\lambda^2$. There exists a second stable solution if $\lambda^2 < 6$ where the scalar field dominates the fluid. It was further shown that these solutions exist if the condition

$$\rho \frac{\rho''}{\rho'^2} - 1 = \rho \frac{d \ln[L(\rho)]}{d\rho} \quad (28)$$

is satisfied [12].

In this Section, we investigate the conditions that allow for scaling solutions in modified gravity to be interpreted as KdV-type solitons. We proceed by rewriting Eq. (28) in the form

$$\frac{\rho''}{\rho} - \frac{3}{2} \frac{\rho'^2}{\rho^2} = \frac{\rho'^2}{\rho^2} \left[\rho \frac{d \ln L}{d\rho} - \frac{1}{2} \right]. \quad (29)$$

Defining a new variable $\sigma = \sigma(\phi)$:

$$\rho = 4\rho_0 \sigma', \quad (30)$$

where ρ_0 is an arbitrary constant, then transforms Eq. (29) into a Schwarzian differential equation:

$$S[\sigma(\phi)] = \left(\frac{\sigma''}{\sigma'} \right)^2 \left[\rho \frac{d \ln L}{d\rho} - \frac{1}{2} \right], \quad (31)$$

where the dependence of the square bracket on σ is implicit.

We now look for particular solutions to Eq. (31) that satisfy the condition that the Schwarzian of σ is constant:

$$S[\sigma(\phi)] = -\frac{\lambda^2}{2}. \quad (32)$$

The discussion of Section 3 implies that the energy density of the universe will satisfy the KdV equation if $\sigma' = (\lambda^2/4) - \sigma^2$. As we saw above, there are two possible solutions when $\lambda^2 > 0$, depending on whether $\sigma' > 0$ or $\sigma' < 0$. (The product $\rho_0 \sigma'$ is assumed implicitly to be positive-definite.)

Eqs. (31) and (32) imply that

$$\left(\frac{\sigma''}{\sigma'} \right)^2 \left[\rho \frac{d \ln L}{d\rho} - \frac{1}{2} \right] = -\frac{\lambda^2}{2}. \quad (33)$$

Since $\sigma'' = -2\sigma\sigma'$, it follows that

$$\left(\frac{\sigma''}{\sigma'} \right)^2 = \lambda^2 - \frac{\rho}{\rho_0}. \quad (34)$$

Consequently, Eq. (33) simplifies to the first-order ODE

$$\frac{d \ln L}{d\rho} = -\frac{1}{2} \frac{1}{\rho_0 \lambda^2 - \rho} \quad (35)$$

and integrating yields the solution

$$L = \left(1 - \frac{\rho}{\rho_0 \lambda^2}\right)^{1/2}, \quad (36)$$

where we have chosen the appropriate branch of the general solution and the constant of integration to ensure that the standard relativistic cosmology is recovered in the low-energy limit $\rho \ll \rho_0 \lambda^2$. The modified Friedmann equation (26) is therefore given by

$$3H^2 = \rho \left(1 - \frac{\rho}{\rho_0 \lambda^2}\right). \quad (37)$$

Finally, the results of Section 3 can be carried over to deduce that the corresponding scaling solutions, when expressed in terms of the energy density, are given by the KdV wave-solutions

$$\rho = \rho_0 \lambda^2 \text{sech}^2(\lambda\phi/2) \quad (38)$$

when $\sigma' > 0$ ($\rho_0 > 0$) and

$$\rho = -\rho_0 \lambda^2 \text{cosech}^2(\lambda\phi/2) \quad (39)$$

when $\sigma' < 0$ ($\rho_0 < 0$).

The Friedmann equation (37) arises in a number of cosmological models that are directly motivated by quantum gravity considerations. When $\rho_0 < 0$, the model corresponds to the Randall-Sundrum braneworld scenario [13], where our observable universe is interpreted as a co-dimension one-brane embedded in five-dimensional, Z_2 symmetric anti-de Sitter space. The coefficient $\rho_0 \lambda^2$ is determined by the tension of the brane [13]. On the other hand, the case where $\rho_0 > 0$ results in the Friedmann equation for the Shtanov-Sahni (S-S) bouncing braneworld [14]. In this model, the universe is again interpreted as a one-brane embedded in a five-dimensional spacetime sourced by a bulk cosmological constant, but the extra fifth dimension is now assumed to be timelike.

Furthermore, the Friedmann equation (37) arises generically in loop quantum cosmological (LQC) scenarios when $\rho_0 > 0$ [15]. In LQC models, the quadratic corrections in the energy density originate from non-perturbative quantum geometric corrections and become important at high energy scales. Indeed, in such a framework, $\rho_0 \lambda^2$ determines a critical density

$$\rho_0 \lambda^2 = \rho_{\text{crit}} = \frac{\sqrt{3}}{16\pi^2 \gamma^3} \rho_{\text{Pl}}, \quad (40)$$

where ρ_{Pl} is the Planck density and $\gamma \approx 0.2375$ is the Immirzi parameter [15].

In effect, therefore, by focusing on the KdV equation we have arrived at three different cosmological models that are all inspired by quantum gravity effects. The corresponding scaling solutions (38) and (39) are those found previously in [12, 16]. We may now interpret these solutions as wave solutions of the KdV equation. The

non-singular soliton solution reflects the non-singular nature of the S-S and LQC scaling solutions. Due to the nature of the quantum corrections, the universe collapses from infinity ($\phi \rightarrow -\infty$), undergoes a non-singular bounce at $\rho = \rho_{\text{crit}}$ ($\phi = 0$), and then expands to infinity ($\phi \rightarrow +\infty$). From the point of view of a stationary laboratory ‘observer’, such dynamics would be equivalent to the time-dependence of the soliton amplitude as the wave propagates (modulo the appropriate relationship between cosmic and laboratory times). The cosmic bounce corresponds to the passing of the peak of the wave. In the LQC scenario, the ‘speed’ of the wave is parametrised by the fractional energy density of the scalar field and the barotropic index of the fluid, $\lambda^2 = 3\gamma/\Omega_\phi$.

V. DISCUSSION

In the present work, it has been shown for the first time that the KdV equation arises in a number of important cosmological scenarios, including the inflationary universe, the cyclic universe, loop quantum cosmology and braneworld cosmology. In each model, cosmological solutions can be reinterpreted as wave-like solutions to the KdV equation, and this allows for analogies to be drawn between cosmic dynamics and wave propagation.

For example, in the inflationary scenario, we have found that the ODE determining the spectral index of the density perturbation spectrum generated during single-field, slow-roll inflation is closely connected to both the Schwarzian derivative operator and the KdV wave equation. In principle, this allows for the full family of inflationary models that generate a constant spectral index to be classified in a very straightforward manner in terms of solutions to the KdV equation.

In the region of observational parameter space $r < 8(1 - n_s)$, a formal analogy was established between the non-singular KdV soliton and the inflating universe. In such a correspondence, the scalar field plays the role of a wave-like coordinate, the speed of the soliton is determined by the value of the spectral index and the amplitude of the soliton is parametrized by the energy density of the universe. Due to the inflationary/cyclic universe duality, similar conclusions hold for the simplest version of the cyclic universe scenario, although in this case the amplitude of the soliton is related to the scale factor of the universe.

The general conditions for scaling solutions in a class of modified cosmological models sourced by a scalar field and a perfect fluid were considered. Requiring that the integral of the cosmic energy density (with respect to the scalar field) has a constant Schwarzian derivative led naturally to a modified Friedmann equation that arises generically in the Randall-Sundrum and Shtanov-Sahni braneworld models [13, 14] and loop quantum cosmology scenarios [15]. In all cases, the corrections to the Friedmann equation are quadratic in the energy density. For such models, the scaling solutions may be interpreted

as wave solutions to the KdV equation, where the cosmic energy density is again analogous to the soliton wave amplitude.

Finally, it is worth remarking that the KdV equation is closely related to the non-linear Schrödinger (NLS) equation. This equation admits solutions that determine the propagation of solitons in Bose-Einstein condensates [17, 18]. A Bose-Einstein condensate is the ground state of a gas of N interacting bosons trapped by an external potential. In the limit where the interaction between the atoms is sufficiently weak, the mean-field approximation may be employed. In this case, the macroscopic wavefunction for the condensate, ψ , satisfies the Gross-Pitaevskii (GP) equation [19]:

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\nabla^2\psi + V(\mathbf{x},t)\psi + g|\psi|^2\psi, \quad (41)$$

where $V(\mathbf{x},t)$ is the trapping potential and m is the mass of the atoms forming the condensate. The scattering coefficient is given by $g = 4\pi\hbar^2 Na/m$, where a is the (s-wave) scattering length.

By employing sufficiently anisotropic trapping potentials, it is possible to reduce the condensate to a quasi-one-dimensional configuration. Typically, the potential is given by $V(x) = \Omega^2 x^2/2$, where the trap strength $\Omega \ll 1$. To a first-approximation, therefore, the potential can be ignored. In this limit, the condensate becomes homogeneous and the GP equation is identical to the integrable non-linear Schrödinger equation:

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\partial_x^2\psi + g|\psi|^2\psi. \quad (42)$$

The defocusing NLS equation (42), where $g > 0$, supports dark soliton solutions [17]. A dark soliton is an envelope excitation characterized by a dip in the ambient density and a phase jump across the density minimum. Such solitons have been observed in a variety of Bose-Einstein condensates in recent years (for a recent review, see [20]). The solution is given by [18]

$$\begin{aligned} |\psi|^2 &= n \left(1 - |\psi_{\text{ds}}|^2\right) \\ |\psi_{\text{ds}}|^2 &= \left(1 - \frac{v^2}{c^2}\right) \text{sech}^2 \left[\sqrt{1 - \left(\frac{v}{c}\right)^2} \frac{(x - vt)}{\xi} \right] \end{aligned} \quad (43)$$

where n is the background density, v is the speed of the soliton, $(x - vt)$ is its position and $c = \sqrt{ng/m}$ is the

sound speed in the condensate. The spatial extent of the soliton is characterized by the healing length $\xi = \hbar/\sqrt{mng} = 1/\sqrt{4\pi na}$. In general, the speed of the soliton is bounded from above by the sound speed, $v < c$.

The density profile of the dark soliton (43) corresponds precisely to the energy density of the inflationary universe (19) and the LQC scaling solution (38). On dimensional grounds, we can make the identification

$$\frac{\phi}{m_{\text{P}}} \longleftrightarrow (x - vt)\sqrt{na} \quad (44)$$

and view the scalar field as a wave-like coordinate. Modulo a constant of proportionality, the spectral index may then be identified with the speed of the soliton:

$$1 - n_s \longleftrightarrow \left(1 - \frac{v^2}{c^2}\right), \quad (45)$$

whereas in the LQC model, the speed is proportional to the kinematic parameters:

$$\frac{\gamma}{\Omega_\phi} \longleftrightarrow \left(1 - \frac{v^2}{c^2}\right). \quad (46)$$

The deviation of the spectral index away from the scale-invariant perturbation spectrum is proportional to the speed of the soliton relative to the condensate sound speed. The maximal speed of the soliton is attained in the limit of the scale-invariant spectrum, $n_s \rightarrow 1$. For the LQC scaling solution, the maximal speed corresponds to an equation of state $p = -\rho$ for the fluid, which is the limit of a cosmological constant.

The above analogies are not intended to be precise, but they do nonetheless suggest a new link between gravitational and non-gravitational systems might be established through the KdV equation. It would be interesting to formalise such analogies further to establish kinematic correspondences between cosmology and condensed matter physics.

Acknowledgements

We thank R. Tavakol for numerous helpful discussions.

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- [1] D. J. Korteweg and H. de Vries, *Philosophical Mag.* **39**, 422 (1895).
 - [2] A. C. Scott, *The Nonlinear Universe*, (Springer-Verlag, Berlin, 2007).
 - [3] H. C. Rosu, *Mod. Phys. Lett. A* **17**, 667 (2002), arXiv:gr-qc/0111002.

- [4] P. D. Lax, *Comm. Pure Appl. Math.* **21**, 467 (1968).
- [5] E. Hille, *Ordinary Differential Equations in the Complex Domain*, (Wiley, New York, 1976).
- [6] J. E. Lidsey, A. R. Liddle, E. W. Kolb, E. J. Copeland, T. Barreiro and M. Abney, *Rev. Mod. Phys.* **69**, 373 (1997).
- [7] D. H. Lyth and A. R. Liddle, *The Primordial Density*

- Perturbation*, (Cambridge University Press, 2009).
- [8] E. Komatsu *et al.*, *Astrophys. J. Suppl.* **192**, 18 (2011), arXiv:1001.4538 [astro-ph.CO].
 - [9] A. Vallinotto, E. J. Copeland, E. W. Kolb, A. R. Liddle and D. A. Steer, *Phys. Rev. D* **69**, 103519 (2004), arXiv:astro-ph/0311005.
 - [10] J. Khoury, P. J. Steinhardt and N. Turok, *Phys. Rev. Lett.* **91**, 161301 (2003), arXiv:astro-ph/0302012.
 - [11] J. E. Lidsey, *Phys. Rev. D* **70**, 041302 (2004), arXiv:gr-qc/0405055.
 - [12] E. J. Copeland, S.-J. Lee, J. E. Lidsey and S. Mizuno, *Phys. Rev. D* **71**, 023526 (2005), arXiv:astro-ph/0410110.
 - [13] L. Randall and R. Sundrum, *Phys. Rev. Lett.* **83**, 4690 (1999), arXiv:hep-th/9906064.
 - [14] Y. Shtanov and V. Sahni, *Phys. Lett. B* **557**, 1 (2002), arXiv:gr-qc/0208047.
 - [15] P. Singh, K. Vandersloot and G. V. Vereshchagin, *Phys. Rev. D* **74**, 043510 (2006), arXiv:gr-qc/0606032.
 - [16] R. M. Hawkins and J. E. Lidsey, *Phys. Rev. D* **63**, 041301 (2001), arXiv:gr-qc/0011060.
 - [17] T. Tsuzuki, *J. Low. Temp. Phys.* **4**, 441 (1971).
 - [18] V. E. Zakharov and A. B. Shabat, *Zh. Eksp. Teor. Fiz.* **64**, 1627 (1973) [*Sov. Phys. JETP* **37**, 823].
 - [19] E. P. Gross, *Nuovo Cim.* **20**, 454 (1961); L. P. Pitaevskii, *Zh. Eksp. Teor. Fiz.* **40**, 646 (1961) [*Sov. Phys. JETP* **13**, 451].
 - [20] D. J. Frantzeskakis, arXiv:1004.4071 [cond-mat.quant-gas].